Exercício 76

$X \sim BN(k,p)$ isto é $f(x\,|\,k,p)=\binom{x-1}{k-1}p^k(1-p)^{k-x}\quad x=k,k+1,\cdots\quad 0<p<1$

Desenvolvimento em série de McLaurin: $f(x)=\sum_{r=0}^{\infty} f^{(r)}(0)\dfrac{x^r}{r!}$ com $f^{(0)}(x)=f(x)$

a)

$h(q)=(1-q)^{-k}\quad 0<q<1$

$$
\begin{aligned}
h'(q) &= k(1-p)^{-(k+1)} \\
&= \sum_{s=0}^{\infty} h^{(s+1)}(0)\frac{q^s}{s!} && \text{A função inicial já é a 1ª derivada} \\
&= \sum_{s=0}^{\infty} \frac{(k+s+1-1)!}{(k-1)!}\frac{q^s}{s!} && \text{ver expressão geral da derivada de ordem } s+1 \text{ no ponto } 0
\end{aligned}
$$

Logo

$$
\begin{aligned}
&= \sum_{r=1}^{\infty} \frac{(k+r-1)!}{(k-1)!}\frac{q^{r-1}}{(r-1)!} && \text{fazendo } r=s+1 \\
&= \sum_{r=1}^{\infty} \frac{(k+r-1)!}{(k-1)!}\frac{r\,q^{r-1}}{r!} \\
&= \sum_{r=1}^{\infty} \binom{k+r-1}{k-1} r\,q^{r-1}
\end{aligned}
$$

A expressão geral das derivadas de $h(q)$ vem

$$h^{(1)}(q)=k(1-q)^{-(k+1)}$$

$$h^{(2)}(q)=k(k+1)(1-q)^{-(k+2)}=\frac{(k+1)!}{(k-1)!}(1-q)^{-(k+2)}=\frac{(k+2-1)!}{(k-1)!}(1-q)^{-(k+2)}$$

$$h^{(3)}(q)=k(k+1)(k+2)(1-q)^{-(k+3)}=\frac{(k+2)!}{(k-1)!}(1-q)^{-(k+3)}=\frac{(k+3-1)!}{(k-1)!}(1-q)^{-(k+3)}$$

$\cdots$

$$h^{(s)}(q)=k(k+1)\cdots(k+s-1)(1-q)^{-(k+s)}=\frac{(k+s-1)!}{(k-1)!}(1-q)^{-(k+s)}$$

b)

$$
\begin{aligned}
E(X-k) &= \sum_{x=k}^{\infty}(x-k)\binom{x-1}{k-1}p^k(1-p)^{x-k} \\
&= \sum_{x=k+1}^{\infty}(x-k)\binom{x-1}{k-1}p^k(1-p)^{x-k} && \text{já que } (x-k)=0 \text{ quando } x=k \\
&= \sum_{r=1}^{\infty} r\binom{r+k-1}{k-1}p^k(1-p)^r && \text{fazendo r} = x-k \\
&= p^k(1-p)\sum_{r=1}^{\infty} r\binom{r+k-1}{k-1}(1-p)^{r-1}
\end{aligned}
$$

$$E(X-k) = p^k\, q\sum_{r=1}^{\infty} r\binom{r+k-1}{k-1}q^{r-1} \qquad \text{fazendo } q=1-p$$

$$= p^k\, qk(1-q)^{-(k+1)}$$

Logo

$$E(X) = E(X-k)+k = k + p^k\, qk(1-q)^{-(k+1)}$$

$$= k\left(1+p^k\, q(1-q)^{-(k+1)}\right) = k\left(1+\frac{p^k\, q}{p^{k+1}}\right) \qquad \text{recordando que } q=1-p \text{ isto é } p=1-q$$

$$= k\left(1+\frac{q}{p}\right) = k\left(\frac{p+q}{p}\right) = \frac{k}{p}$$

c)

$$h''(q) = k(k+1)(1-q)^{-(k+2)} \qquad \text{ver alínea a)}$$

$$= \sum_{s=0}^{\infty} h^{(s+2)}(0)\frac{q^s}{s!} \qquad \text{A função inicial já é a 2ª derivada}$$

$$= \sum_{s=0}^{\infty} \frac{(k+s+2-1)!}{(k-1)!}\frac{q^s}{s!} \qquad \text{ver expressão geral da derivada de ordem } s+2 \text{ no ponto } 0$$

$$= \sum_{r=2}^{\infty} \frac{(k+r-1)!}{(k-1)!}\frac{q^{r-2}}{(r-2)!} \qquad \text{fazendo } r=s+2$$

$$= \sum_{r=2}^{\infty} \frac{(k+r-1)!}{(k-1)!}\frac{r\,(r-1)\,q^{r-2}}{r!} \qquad \text{multiplicando e dividindo por } r! \text{ e simplificando}$$

$$= \sum_{r=2}^{\infty} \binom{k+r-1}{k-1} r\,(r-1)\,q^{r-2}$$

d)

$$E\big((X-k)(X-k-1)\big) = \sum_{x=k}^{\infty} (x-k)\,(x-k-1)\binom{x-1}{k-1}p^k\,(1-p)^{x-k}$$

$$= \sum_{x=k+2}^{\infty} (x-k)(x-k-1)\binom{x-1}{k-1}p^k\,(1-p)^{x-k} \qquad \text{eliminar os 2 1os termos que são nulos}$$

$$= \sum_{r=2}^{\infty} r\,(r-1)\binom{r+k-1}{k-1}p^k\,(1-p)^r \qquad \text{fazendo r} = x-k$$

$$= p^k\,(1-p)^2\sum_{r=1}^{\infty} r(r-1)\binom{r+k-1}{k-1}(1-p)^{r-2}$$

$$= p^k\,q^2\sum_{r=1}^{\infty} r(r-1)\binom{r+k-1}{k-1}q^{r-2} \qquad \text{fazendo } q=1-p$$

$$= p^k\,q^2k(k+1)(1-q)^{-(k+2)} \qquad \text{utilizando alínea c}$$

$$= q^2k(k+1)\,p^{-2} \qquad \text{fazendo } q=1-p \text{ e simplificando}$$

e) Notando que

$$E\left((X-k)(X-k-1)\right)=E\left((X-k)\left((X-k)-1\right)\right)=E\left(\left(X-k\right)^2-\left(X-k\right)\right)$$
$$=E(X-k)^2-E(X-k)$$

Vem

$$\begin{aligned}\operatorname{var}(X-k)&=E(X-k)^2-\left(E(X-k)\right)^2\\&=E(X-k)^2-E(X-k)-\left(E(X-k)\right)^2+E(X-k)\\&=E\left(X-k)(X-k-1)\right)-\left(E(X-k)\right)^2+E(X-k)\\&=q^2p^{-2}k(k+1)-\left(p^kqk\,p^{-(k+1)}\right)^2+p^kqk\,p^{-(k+1)}\\&=q^2p^{-2}k(k+1)-q^2p^{-2}k^2+qk\,p^{-1}\\&=\frac{q^2k(k+1)-q^2k^2+q\,pk}{p^2}=\frac{q^2k^2+q^2k-q^2k^2+q\,(1-q)k}{p^2}\\&=\frac{q^2k+qk-q^2k}{p^2}=\frac{qk}{p^2}\end{aligned}$$

Ora como $\operatorname{var}(X-k)=\operatorname{var}(X)$ obtém-se $\operatorname{var}(X)=\dfrac{qk}{p^2}$.